INTERMODAL FOUR-WAVE MIXING IN A HIGHER-ORDER-MODE FIBER

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ABSTRACT

Much effort has gone into the study of silica and its material properties due to its importance in long distance communication. Especially, the propagation of optical pulses in single-mode fibers and their interplay with nonlinear effects have been modeled with good accuracy. This thesis first presents a derivation of a frequency-domain Generalized Nonlinear Shrödinger Equation that has been used in the simulation of nonlinear phenomena relating to ultrashort pulses in single-mode fibers. It then develops an even more general approach and shows in detail how it can be used to successfully model the results of an experiment in which high-efficiency intermodal four-wave mixing in a higher-order-mode (or few-mode) fiber was observed. A formulation that can model interactions between vector-polarized modes is also covered.

BIOGRAPHICAL SKETCH

The author spent his formative years in Islamabad, Pakistan. After graduating with a degree in Mathematics, Operations Research, Statistics and Economics from the University of Warwick (England) and working in a professional services role for a couple of years, he entered Cornell University for graduate study in Applied Physics in Fall 2011.

بسم الله الرحمن الرحيم

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INTRODUCTION

1.1 Nonlinear polarization

In the classical treatment of interaction between light and matter, Maxwell's equations are considered to describe with reasonable accuracy both electromagnetic radiation and the response of the macroscopic medium being considered. Specifically, the spatial variation of the electric field over molecular distance scales is ignored.

The basic approach described above is also useful in describing a range of nonlinear optical phenomena, first discovered by Franken [2] in 1961 in the form of second-harmonic generation and theoretically covered soon after in a paper by Armstrong et al. [3]. Thus the following equations, which assume a nonmagnetic material having no free charges, will serve as the launch pad for further investigation.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.1a}$$

$$\nabla \cdot \mathbf{D} = \rho_f = 0, \tag{1.1b}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} = \frac{\partial \mathbf{D}}{\partial t}, \tag{1.1c}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.1d}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \mathbf{P}_{\mathrm{L}} + \mathbf{P}_{\mathrm{NL}}. \tag{1.1e}$$

Note that the quantities in bold above are all time-domain expressions; they are functions of time t and space $\mathbf{r}$. The displacement term $\mathbf{D}$ has been written in the most general form to emphasize the nonlinearity, and $\mathbf{H} = \mathbf{B}/\mu_0$. An additional complication to nonlinearity is anisotropy, which occurs in media where the polarization response to a given applied field varies with direction. The total polarization can be described as

$$\mathbf{P} = \underbrace{\epsilon_0 \boldsymbol{\chi}^{(1)}\mathbf{E}}_{\mathbf{P}_{\mathrm{L}}=\mathbf{P}^{(1)}} + \underbrace{\overbrace{\epsilon_0 \boldsymbol{\chi}^{(2)}\mathbf{E}\mathbf{E}}^{\mathbf{P}^{(2)}} + \overbrace{\epsilon_0 \boldsymbol{\chi}^{(3)}\mathbf{E}\mathbf{E}\mathbf{E}}^{\mathbf{P}^{(3)}} + \ldots}_{\mathbf{P}_{\mathrm{NL}}}, \tag{1.2}$$

where $\boldsymbol{\chi}^{(i)}$ is the i^{th} order susceptibility tensor. Note that the order of the tensor itself is $i+1$, which implies that

$$P_i^{(1)} = \epsilon_0 \sum_j \chi_{ij}^{(1)} E_i. \tag{1.3}$$

The second order nonlinearity can also be expanded in summation notation to give

$$\begin{aligned} P_i^{(2)}\hat{\mathbf{e}}_i &= \epsilon_0 \chi_{ijk}^{(2)}\, \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \cdot \left(E_l \hat{\mathbf{e}}_l\right) \cdot \left(E_m \hat{\mathbf{e}}_m\right) && \text{(1.4a)} \\ &= \epsilon_0 \chi_{ijk}^{(2)}\, \hat{\mathbf{e}}_i \left(E_l \delta_{lk}\right)\left(E_m \delta_{mj}\right) && \text{(1.4b)} \\ &= \epsilon_0 \chi_{ijk}^{(2)} E_j E_k\, \hat{\mathbf{e}}_i, && \text{(1.4c)} \\ \Longrightarrow P_i^{(2)} &= \epsilon_0 \sum_j \sum_k \chi_{ijk}^{(2)} E_j E_k. && \text{(1.4d)} \end{aligned}$$

Higher order nonlinearities can be expanded likewise. If the material is symmetric, as is the case with silica, $\boldsymbol{\chi}^{(2)}$ can be ignored and the isotropy of silica simplifies the tensor $\boldsymbol{\chi}^{(3)}$ to a scalar value. Nonlinearities of order greater than three are too weak to be of significance. Indeed, $\mathbf{P}_{\mathrm{NL}}$ will itself be considered a perturbation in the treatment to follow later.

1.2 Time and frequency-domain representations

In later sections, the distinction between frequency and time-domain versions of fields, polarizations and susceptibilities will be crucial. This necessitates the introduction of notation introduced in this section.

If the polarization response $P^{(1)}(t)$ to an applied field $E(t)$ is non-instantaneous then the polarization will be written as

$$P^{(1)}(t) = \epsilon_0 \int \mathrm{d}\tau\, \chi^{(1)}(\tau) E(t-\tau). \tag{1.5}$$

Here $P^{(1)}(t)$ is the polarization at time t as a function of the electric field at earlier times and of the response function $\chi^{(1)}(\tau)$ (which is zero for values $\tau < 0$ to ensure causality).

Defining $\tilde{E}(\omega)$ such that

$$E(t) = \mathscr{F}^{-1}\left\{\tilde{E}(\omega)\right\} = \int \frac{\mathrm{d}\omega}{2\pi}\, \tilde{E}(\omega) e^{-i\omega t},$$

we can write

$$P^{(1)}(t) = \epsilon_0 \int \mathrm{d}\tau\, \chi^{(1)}(\tau) \int \frac{\mathrm{d}\omega}{2\pi}\, \tilde{E}(\omega) e^{-i\omega(t-\tau)} \tag{1.6a}$$

$$= \epsilon_0 \int \frac{\mathrm{d}\omega}{2\pi} \left[\int \mathrm{d}\tau\, \chi^{(1)}(\tau) e^{i\omega\tau}\right] \tilde{E}(\omega) e^{-i\omega t}. \tag{1.6b}$$

With

$$\tilde{\chi}^{(1)}(\omega) \equiv \int \mathrm{d}\tau\, \chi^{(1)}(\tau) e^{i\omega\tau}, \tag{1.7}$$

we can simplify this to

$$P^{(1)}(t) = \epsilon_0 \int \frac{\mathrm{d}\omega}{2\pi}\, \tilde{\chi}^{(1)}(\omega) \tilde{E}(\omega) e^{-i\omega t} \quad \Longrightarrow \quad \tilde{P}^{(1)}(\omega) = \epsilon_0 \tilde{\chi}^{(1)}(\omega) \tilde{E}(\omega). \tag{1.8}$$

In case the response is instantaneous the response function would be of the form $\chi^{(1)}(\tau) = \chi^{(1)}_{xx}\delta(\tau)$, where the entire time-dependence has been restricted to the delta function and $\chi^{(1)}_{xx} \in \mathbb{R}$, which would give the polarization as

$$P^{(1)}(t) = \epsilon_0 \int \mathrm{d}\tau\, \chi^{(1)}_{xx} E(t-\tau)\delta(\tau) e^{i\omega\tau} = \epsilon_0 \chi^{(1)}_{xx} E(t).$$

In the frequency-domain, this would imply a relation similar to the second equation in eq. (1.8) but with $\tilde{\chi}^{(1)}(\omega)$ replaced by $\chi^{(1)}_{xx}$.

1.2.1 Raman scattering

An important nonlinear effect relating to ultrashort pulses is Raman scattering. Raman scattering refers to the coupling of pump light with non-propagating internal modes (vibrational, rotational or electronic) of the medium. For instance, nonlinear index [4] changes can result from optically induced nucleic motion that modify the polarizability. The most common type is Stokes scattering which leads to generation of light that is downshifted in frequency. Studies on the material properties of silica [5–12] have also characterized this phenomenon.

Further, for optical frequencies much lower than optical transitions in transparent media, the electronic response is nearly instantaneous. This is not the case with non-electronic responses, which may have a time-delayed contribution to the nonlinear index.

In similar vein to the first-order time-domain function described in detail above, we have

$$P_i^{(3)}(\mathbf{r},t) = \epsilon_0 \int \mathrm{d}t_1 \int \mathrm{d}t_2 \int \mathrm{d}t_3\, \chi_{ijkl}(t,t_1,t_2,t_3) E_j(\mathbf{r},t_1) E_k(\mathbf{r},t_2) E_l(\mathbf{r},t_3). \quad (1.9)$$

Using an adiabatic quantum-mechanical approach and considering strictly optical nonlinear effects leads to an expression of the form [11]

$$\begin{aligned} P_i^{(3)}(\mathbf{r},t) = \epsilon_0 \Big\{ & \sigma_{ijkl} E_j(\mathbf{r},t_1) E_k(\mathbf{r},t_2) E_l(\mathbf{r},t_3) \\ & + E_j(\mathbf{r},t_1) \int \mathrm{d}\tau\, d_{ijkl}(t-\tau) E_k(\mathbf{r},\tau) E_l(\mathbf{r},\tau) \Big\}, \end{aligned} \quad (1.10)$$

where

$$\sigma_{ijkl} \equiv \sigma \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \quad (1.11)$$

$$d_{ijkl}(t) \equiv d_a(t)\delta_{ij}\delta_{kl} + \frac{1}{2} d_b(t) \left(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\right). \quad (1.12)$$

This gives

$$\mathbf{P}^{(3)}(\mathbf{r},t) = \epsilon_0 \Bigg\{ 3\sigma\mathbf{E}(\mathbf{r},t)\big|\mathbf{E}(\mathbf{r},t)\big|^2 + \mathbf{E}(\mathbf{r},t)\int \mathrm{d}\tau\, d_a(t-\tau)\big|\mathbf{E}(\mathbf{r},t)\big|^2 + \int \mathrm{d}\tau\, d_b(t-\tau)\big|\mathbf{E}(\mathbf{r},t)\big|^2\mathbf{E}(\mathbf{r},\tau) \Bigg\}, \quad (1.13)$$

For our purposes, we use eq. (1.9) and take fields of the the linearly polarized form

$$\mathbf{E}(\mathbf{r},t) \equiv \hat{\mathbf{p}}\,\mathcal{E}(\mathbf{r},t)e^{-i\omega_0 t} + \text{c.c.}, \quad (1.14a)$$

$$\mathbf{P}^{(3)}(\mathbf{r},t) \equiv \hat{\mathbf{p}}\,\mathcal{P}^{(3)}(\mathbf{r},t)e^{-i\omega_0 t} + \text{c.c.} \quad (1.14b)$$

Ignoring third-harmonic generation due to a general lack of phase-matching, the third-order nonlinear polarization is specified using

$$\mathcal{P}^{(3)}(\mathbf{r},t) = \epsilon_0 3\chi^{(3)}_{xxxx}\mathcal{E}(\mathbf{r},t)\int \mathrm{d}\tau\, R(t-\tau)\big|\mathcal{E}(\mathbf{r},t)\big|^2, \quad (1.15)$$

where the Raman response function is modeled as [13]

$$R(t) \equiv (1-f_\mathrm{R})\delta(t) + f_\mathrm{R}h_\mathrm{R}(t), \quad (1.16a)$$

$$h_\mathrm{R}(\tau) \equiv \tau_1\left(\tau_1^{-2}+\tau_2^{-2}\right)e^{-t/\tau_2}\sin(t/\tau_1), \quad (1.16b)$$

with $f_\mathrm{R} = 0.18$, $\tau_1 = 12.2$ fs and $\tau_2 = 32$ fs. h_R is used in practice as a simplified approximation (the single-Lorentzian model) to the integrals in eq. (1.13), which can be more complex, and f_R is used to weight the electronic and nuclear responses.

1.3 Linear and nonlinear permittivity and absorption

It is important to specify clearly the time and frequency-domain relations between permittivity and susceptibility, as this will allow a clean transition to a general nonlinear wave equation later on. Absorption must also be considered since, realistically, energy is not conserved as it travels through the medium; no signal propagates indefinitely. Working with eq. (1.1e), if the displacement is defined as

$$\mathbf{D} \equiv \epsilon \mathbf{E}, \tag{1.17}$$

the time-domain permittivity can be written

$$\epsilon = \epsilon_0(\epsilon_{\mathrm{L}} + \epsilon_{\mathrm{NL}}) = \epsilon_0 \mathrm{Re}(\epsilon_{\mathrm{L}}) + \Delta\epsilon = \epsilon_0 \left[\mathrm{Re}(\epsilon_{\mathrm{L}}) + \mathrm{Im}(\epsilon_{\mathrm{L}}) + \epsilon_{\mathrm{NL}}\right], \tag{1.18}$$

where $\Delta\epsilon = i\,\mathrm{Im}(\epsilon_{\mathrm{L}}) + \epsilon_{\mathrm{NL}}$.

The linear (relative) permittivity and first order susceptibility are related by the definition

$$\epsilon_{\mathrm{L}} \equiv 1 + \chi^{(1)}. \tag{1.19}$$

As a result, the frequency-domain definition of the linear permittivity becomes

$$\tilde{\epsilon}_{\mathrm{L}}(\omega) \equiv 1 + \tilde{\chi}^{(1)}(\omega) = 1 + \tilde{\chi}^{(1)}_{\mathrm{Re}}(\omega) + i\,\tilde{\chi}^{(1)}_{\mathrm{Im}}(\omega), \tag{1.20}$$

where the subscripts Re and Im have used to denote the real and imaginary parts of $\tilde{\chi}^{(1)}(\omega)$.

At this stage we define the relation between permittivity, linear index of refraction and frequency dependent absorption by

$$\tilde{\epsilon}_{\mathrm{L}}(\omega) \equiv \left(\tilde{n}_{\mathrm{L}}(\omega) + i\,\frac{c\tilde{\alpha}(\omega)}{2\omega}\right)^2. \tag{1.21}$$

The last two equations now give a relationship between the susceptibility, linear index and absorption:

$$\sqrt{1 + \tilde{\chi}^{(1)}(\omega)} = \tilde{n}_{\mathrm{L}}(\omega) + i\,\frac{c\tilde{\alpha}(\omega)}{2\omega}. \tag{1.22}$$

Assuming that $\tilde{\chi}^{(1)}_{\mathrm{Im}}$ is small in comparison to $1 + \tilde{\chi}_{\mathrm{Re}}$ (the low loss approximation) we may expand the square root term to give

$$\sqrt{1+\tilde{\chi}^{(1)}(\omega)} \approx \left(1+\tilde{\chi}^{(1)}_{\mathrm{Re}}(\omega)\right)^{1/2} + \frac{i}{2}\frac{\tilde{\chi}^{(1)}_{\mathrm{Im}}(\omega)}{1+\tilde{\chi}^{(1)}_{\mathrm{Re}}(\omega)} \tag{1.23a}$$

$$= \tilde{n}_{\mathrm{L}}(\omega) + i\,\frac{c\tilde{\alpha}(\omega)}{2\omega}, \tag{1.23b}$$

giving the relations

$$\tilde{n}_{\mathrm{L}}(\omega) \equiv \sqrt{1+\tilde{\chi}^{(1)}_{\mathrm{Re}}(\omega)}, \tag{1.24a}$$

$$\tilde{\alpha}(\omega) \equiv \frac{\omega}{c\tilde{n}_{\mathrm{L}}(\omega)}\tilde{\chi}^{(1)}_{\mathrm{Im}}(\omega). \tag{1.24b}$$

Finally, the approximations above also allow writing

$$1+\tilde{\chi}^{(1)}(\omega) = \tilde{n}^2_{\mathrm{L}}(\omega) + i\,\frac{c\tilde{\alpha}(\omega)\tilde{n}_{\mathrm{L}}(\omega)}{\omega}. \tag{1.25}$$

1.4 Dispersive pulse propagation

Due to the effects described earlier, a pulse can alter properties of the material it traverses and consequently, itself experience the effect of those changes. In addition, chromatic dispersion – each frequency component experiencing a difference refractive index and hence traveling at a different velocity – is inevitable. Considerable attention has been given to modeling these properties [14–18] with most formulations using envelope representations, the basics of which are covered in this section.

Before presenting a frequency-domain derivation that describes pulse propagation in detail and incorporates higher order effects, an example of dispersive pulse propagation will be given below to introduce the key ideas. This will also inform our choice of ansatz solution to the electromagnetic field.

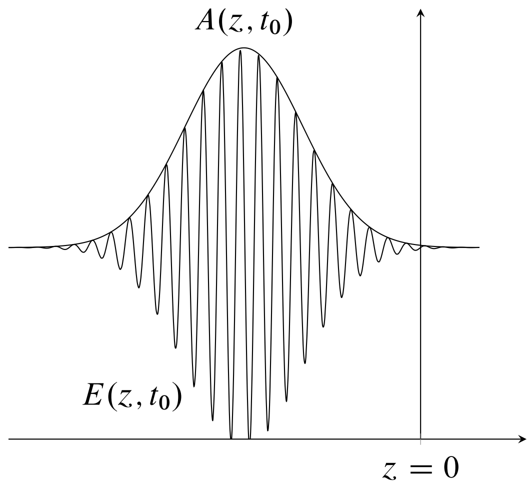

Figure 1.1. A Gaussian pulse. The dispersive region lies to the right of the y-axis.

In the time-domain a pulse can be described in the form

$$E(z,t) = A(z,t)e^{i(\beta_0 z - \omega_0 t)}, \tag{1.26}$$

which describes a carrier wave with frequency ω_0 and wavenumber β_0 modulated by an envelope function $A(z,t)$ (see Figure 1.1) . This is valid for plane waves, and no transverse intensity profile is assumed. In a medium where propagation is not dispersive, we can specify the wave fully by $E(z,t)$ given above since each frequency component travels at the same speed ω_0/β_0. The expression $A(0,t)$ would be sufficient to define the incoming waveform in the dispersionless region. However, if the same pulse enters a dispersive medium at location $z = 0$, this will no longer hold. To specify propagation in this case, the Fourier transform of $E(0,t)$ is calculated as (see Appendix A)

$$\mathscr{F}\{E(0,t)\} = \int \mathrm{d}t\ A(0,t)e^{-i\omega_0 t}e^{i\omega t} = \tilde{A}(0, \omega - \omega_0), \tag{1.27}$$

which gives the weight of each spectral component of $E(0,t)$.

Next, each frequency travels at its propagation constant $\beta(\omega)$ and the inverse Fourier transform is taken to yield the shape of the pulse in time-domain after distance z:

$$E(z,t) = \int \frac{\mathrm{d}\omega}{2\pi}\, \tilde{A}(0,\omega-\omega_0)e^{i\beta(\omega)z}e^{-i\omega t} \tag{1.28a}$$

$$= \underbrace{\int \frac{\mathrm{d}\omega}{2\pi}\, \underbrace{\tilde{A}(0,\omega)e^{i[\beta(\omega+\omega_0)-\beta_0]z}}_{\tilde{A}(z,\omega)}e^{-i\omega t}}_{A(z,t)}\, e^{i(\beta_0 z-\omega_0 t)}. \tag{1.28b}$$

The calculation of $\beta(\omega)$ would depend on the material and, in the case of propagation in a waveguide, the associated boundary conditions.

CHAPTER 2

SINGLE-MODE FORMULATION

2.1 Frequency-domain Generalized Nonlinear Schrödinger Equation

The standard nonlinear Schrödinger equation is the common tool used for modeling ultrashort pulse propagation, and its derivation has been presented in a number of popular textbooks and theses in the subject area of nonlinear and ultrafast optics [13]. In this section we will derive a generalized frequency-domain formulation which has been successfully used in the simulation of supercontinuum generation [19] and recently for intermodal four-wave mixing [1]. As we shall see, the frequency dependence of the transverse field will be retained to first order, rather than neglecting it altogether.

The foundations laid in the previous chapter will be used here, and the first step is to take the curl of eq. (1.1a) while using eq. (1.1c):

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \frac{\partial}{\partial t} \left(\nabla \times \mathbf{H} \right) = -\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2}. \tag{2.1}$$

Recognizing the identity $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$, taking $\nabla(\nabla \cdot \mathbf{E}) \approx 0$ (see text following eq. (2.2)) gives

$$\nabla^2 \mathbf{E} = \mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2}. \tag{2.2}$$

Further to the approximation above, recalling the definition of $\mathbf{D}$ in eq. (1.1e) and noting from eq. (1.1b) that its divergence is zero – in the case of (source-free) isotropic and nonlinear media we have

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \nabla \cdot (\mathbf{E} + \boldsymbol{\chi}^{(1)} \mathbf{E}) \epsilon_0 + \nabla \cdot \mathbf{P}_{\mathrm{NL}} \tag{2.3a}$$

$$= \epsilon_0 \nabla \cdot \mathbf{E} + \nabla \cdot (\mathbf{P}_{\mathrm{NL}} + \epsilon_0 \boldsymbol{\chi}^{(1)} \mathbf{E}) = 0 \tag{2.3b}$$

$$\not\Rightarrow \nabla \cdot \mathbf{E} = 0.$$

We can see that both nonlinearity and anisotropy do not allow $\nabla \cdot \mathbf{E}$ to vanish. If we assume however, that all second-order derivatives of the components of $\mathbf{E}$ are small, neglecting $\nabla(\nabla \cdot \mathbf{E})$ can be justified[1].

Continuing to work with eq. (2.2), the next step is to expand $\mathbf{D}$ and write

$$\nabla^2 \mathbf{E} = \mu_0 \frac{\partial^2 \mathbf{D}(\mathbf{r}, t)}{\partial t^2} = \mu_0 \frac{\partial^2}{\partial t^2} \left[\epsilon_0 \mathbf{E}(\mathbf{r}, t) + \mathbf{P}_{\mathrm{L}}(\mathbf{r}, t) + \mathbf{P}_{\mathrm{NL}}(\mathbf{r}, t) \right]. \tag{2.4}$$

Now we the take the Fourier transform of the above equation, which gives

$$\nabla^2 \tilde{\mathbf{E}} = -\mu_0 \omega^2 \left[\epsilon_0 \tilde{\mathbf{E}}(\mathbf{r}, \omega) + \tilde{\mathbf{P}}_{\mathrm{L}}(\mathbf{r}, \omega) + \tilde{\mathbf{P}}_{\mathrm{NL}}(\mathbf{r}, \omega) \right]. \tag{2.5}$$

Before continuing to work with eq. (2.5) in its entirety, the linear polarization term can be specified first using the general ideas presented in section 1.2. Further, we assume a linearly polarized electric field and that the resulting polarization response also has the same orientation. We define the first order (linear) polarization as

$$\mathbf{P}_{\mathrm{L}}(\mathbf{r}, t) = \epsilon_0 \int \mathrm{d}\tau \, \chi^{(1)}(\mathbf{r}, \tau) \mathbf{E}(\mathbf{r}, t - \tau) = \epsilon_0 \chi^{(1)}(\mathbf{r}, t) \otimes \mathbf{E}(\mathbf{r}, t), \tag{2.6}$$

where the convolution has been explicitly stated in anticipation of the conversion to frequency-domain. Therefore

$$\tilde{\mathbf{P}}_{\mathrm{L}}(\mathbf{r}, \omega) = \mathscr{F} \left\{ \mathbf{P}_{\mathrm{L}}(t, \mathbf{r}) \right\} = \epsilon_0 \tilde{\chi}^{(1)}(\mathbf{r}, \omega) \tilde{\mathbf{E}}(\mathbf{r}, \omega), \tag{2.7}$$

[1] This is because $\nabla(\nabla \cdot \mathbf{E})$ can be expanded to show that it consists of a sum of the second-order derivatives of E_x, E_y and E_z.

according to the standard rules of Fourier transformation (see Appendix A).

The groundwork of section 1.3 will now be used to introduce some of the earlier variables into eq. (2.5) as follows:

$$
\begin{aligned}
\nabla^2 \tilde{\mathbf{E}} &= -\mu_0 \omega^2 \left[\epsilon_0 \tilde{\mathbf{E}}(\mathbf{r}, \omega) + \epsilon_0 \tilde{\chi}^{(1)}(\mathbf{r}, \omega) \tilde{\mathbf{E}}(\mathbf{r}, \omega) + \tilde{\mathbf{P}}_{\mathrm{NL}}(\mathbf{r}, \omega) \right] && \text{(2.8a)} \\
&= -\mu_0 \omega^2 \left[\epsilon_0 \left(1 + \tilde{\chi}^{(1)} \right) \tilde{\mathbf{E}}(\mathbf{r}, \omega) + \tilde{\mathbf{P}}_{\mathrm{NL}}(\mathbf{r}, \omega) \right] && \text{(2.8b)} \\
&= -\mu_0 \omega^2 \left[\epsilon_0 \left(\tilde{n}_{\mathrm{L}}^2(\mathbf{r}_\perp, \omega) + i \frac{\tilde{\alpha}(\omega) c \tilde{n}_{\mathrm{L}}(\omega)}{\omega} \right) \tilde{\mathbf{E}}(\mathbf{r}, \omega) + \tilde{\mathbf{P}}_{\mathrm{NL}}(\mathbf{r}, \omega) \right] && \text{(2.8c)} \\
&= -\frac{\omega^2 \tilde{n}_{\mathrm{L}}^2(\mathbf{r}_\perp, \omega)}{c^2} \tilde{\mathbf{E}}(\mathbf{r}, \omega) - i \frac{\omega \tilde{\alpha}(\omega) \tilde{n}_{\mathrm{L}}(\mathbf{r}_\perp, \omega)}{c} \tilde{\mathbf{E}}(\mathbf{r}, \omega) \\
&\quad - \mu_0 \omega^2 \tilde{\mathbf{P}}_{\mathrm{NL}}(\mathbf{r}, \omega), && \text{(2.8d)}
\end{aligned}
$$

where the definitions in eqs. (1.22)-(1.25) have been employed.

Progress beyond this point will be facilitated by specification of the electric field. Keeping in mind the discussion in section 1.4, and without straying far from convention found in literature, we define it as[2]

$$\mathbf{E}(\mathbf{r}, t) = \hat{\mathbf{p}} \, \mathcal{E}(x, y, z, t) e^{-i\omega_0 t} + \text{c.c.}, \tag{2.9a}$$

where[3]

$$
\begin{aligned}
\mathcal{E}(x, y, z, t) e^{-i\omega_0 t} &= \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \tilde{E}(\mathbf{r}, \omega) e^{-i\omega t} = \int \frac{\mathrm{d}\omega}{2\pi} \tilde{A}(z, \omega) \, F(x, y, \omega) e^{i\left(\beta(\omega) z - \omega t\right)} \\
&= \int \frac{\mathrm{d}\omega}{2\pi} \tilde{A}(z, \omega) \, F(\mathbf{r}_\perp, \omega) e^{i\left(\beta(\omega) z - \omega t\right)}. && \text{(2.9b)}
\end{aligned}
$$

The spectral envelopes for positive and negative frequencies are taken to be well separated for the electric field (and also the nonlinear polarization). $\hat{\mathbf{p}}$ above is a unit vector

[2]Backward propagating waves are neglected.

[3]Similarly $E(\mathbf{r}, t) = \mathcal{E}(x, y, z, t) e^{-i\omega_0 t}$ + c.c. and $\mathscr{F}\{E(\mathbf{r}, t)\} = \tilde{E}(\mathbf{r}, \omega)$. Also note that $\tilde{A}(z, \omega)$ is sometimes written with a frequency shift of ω_0, this will be introduced later through an (optional) variable substitution.

orthogonal to the longitudinal direction $\hat{\mathbf{z}}$. F (the frequency dependent transverse mode profile) and β are obtained as solutions to the linear (unperturbed) eigenvalue equation [13]

$$\nabla_\perp^2 F + \left(\frac{\omega^2 n^2}{c^2} - \beta^2\right) F = 0. \tag{2.10}$$

Working with the above representation allows writing the positive frequency component of the LHS of eq. (2.8) as[4]

$$\begin{aligned}
&\nabla^2 F(\mathbf{r}_\perp, \omega)\tilde{A}(z,\omega)e^{i\beta(\omega)z} \\
&= \left(\nabla_\perp^2 + \frac{\partial^2}{\partial z^2}\right) F(\mathbf{r}_\perp, \omega)\tilde{A}(z,\omega)e^{i\beta(\omega)z} && \text{(2.11a)} \\
&= \tilde{A}(z,\omega)\nabla_\perp^2 F(\mathbf{r}_\perp,\omega)e^{i\beta(\omega)z} + F(\mathbf{r}_\perp,\omega)\frac{\partial^2}{\partial z^2}\left(\tilde{A}(z,\omega)e^{i\beta(\omega)z}\right) && \text{(2.11b)} \\
&= \tilde{A}(z,\omega)\nabla_\perp^2 F(\mathbf{r}_\perp,\omega)e^{i\beta(\omega)z} \\
&\quad + F(\mathbf{r}_\perp,\omega)\left[\frac{\partial^2 \tilde{A}(z,\omega)}{\partial z^2} + 2i\beta(\omega)\frac{\partial \tilde{A}(z,\omega)}{\partial z} - \beta^2(\omega)\tilde{A}(z,\omega)\right] \\
&\quad \times e^{i\beta(\omega)z}. && \text{(2.11c)}
\end{aligned}$$

Similarly, the RHS becomes[5]

$$\begin{aligned}
- F(\mathbf{r}_\perp,\omega)\tilde{A}(z,\omega)e^{i\beta(\omega)z}\left(\frac{\omega^2 \tilde{n}_{\mathrm{L}}^2(\mathbf{r}_\perp,\omega)}{c^2} + i\,\frac{\omega\tilde{\alpha}(\omega)\tilde{n}_{\mathrm{L}}(\mathbf{r}_\perp,\omega)}{c}\right) \\
- \mu_0\omega^2\tilde{\mathcal{P}}_{\mathrm{NL}}(\mathbf{r},\omega). \quad \text{(2.12)}
\end{aligned}$$

Equating both sides, using the spectral version of the slowly-varying envelope/amplitude approximation (SVEA/SVAA)[6]

$$\left|\frac{\mathrm{d}^2\tilde{A}}{\mathrm{d}z^2}\right| \ll 2\beta\left|\frac{\mathrm{d}\tilde{A}}{\mathrm{d}z}\right| \ll \beta^2\left|\tilde{A}\right|, \tag{2.13}$$

[4] $\hat{\mathbf{p}}$ will be ignored in most places since the polarization is understood to be linear.

[5] Bear in mind that $\tilde{\mathcal{P}}_{\mathrm{NL}}$ has incorporated the shift in ω by ω_0, just as in the definition of $\tilde{A}$, so it is not the precise Fourier transform of $\mathcal{P}^{(3)}$ as defined in (1.15).

[6] Also recall that we assumed the time-domain second-order derivatives are negligible in eq. (2.2).

so that the second-order derivative with respect to z can be ignored[7], and rearranging gives

$$\left(\nabla_\perp^2 + \frac{\omega^2 \tilde{n}_\mathrm{L}^2(\mathbf{r}_\perp,\omega)}{c^2} - \beta^2(\omega)\right) F(\mathbf{r}_\perp,\omega)\tilde{A}(z,\omega) + 2i\beta(\omega)F(\mathbf{r}_\perp,\omega)\frac{\partial \tilde{A}(z,\omega)}{\partial z}$$
$$= -i\,\frac{\omega\tilde{\alpha}(\omega)\tilde{n}_\mathrm{L}(\mathbf{r}_\perp,\omega)}{c}F(\mathbf{r}_\perp,\omega)\tilde{A}(z,\omega) - \mu_0\omega^2\tilde{\mathcal{P}}_\mathrm{NL}(\mathbf{r},\omega)e^{-i\beta(\omega)z}. \quad (2.14)$$

The first term in the above equation will vanish because $F(\mathbf{r}_\perp,\omega)$ is the solution of the Helmholtz equation governing propagation in a fiber in the unperturbed case ($\mathbf{P}_\mathrm{NL}(\mathbf{r},t) = 0$), with $\beta(\omega)$ as the mode propagation constant. This gives

$$F(\mathbf{r}_\perp,\omega)\frac{\partial \tilde{A}(z,\omega)}{\partial z} = -\frac{\omega\tilde{\alpha}(\omega)\tilde{n}_\mathrm{L}(\mathbf{r}_\perp,\omega)}{2c\beta(\omega)}F(\mathbf{r}_\perp,\omega)\tilde{A}(z,\omega) + i\,\frac{\mu_0\omega^2}{2\beta(\omega)}\tilde{\mathcal{P}}_\mathrm{NL}(\mathbf{r},\omega)e^{-i\beta(\omega)z}$$
$$= -\frac{\tilde{\alpha}(\omega)}{2}F(\mathbf{r}_\perp,\omega)\tilde{A}(z,\omega) + i\,\frac{\mu_0\omega^2}{2\beta(\omega)}\tilde{\mathcal{P}}_\mathrm{NL}(\mathbf{r},\omega)e^{-i\beta(\omega)z}, \quad (2.15)$$

where in the second step, the additional approximation $\beta(\omega) \approx \omega\tilde{n}_\mathrm{L}/c$ has been made.

The Fourier transformation of the nonlinear polarization term will be similar to the process carried out in eqs. (2.6) and (2.7). Here however, the spectral formulation is slightly more involved. The second-order susceptibility will vanish because of inversion symmetry. The third-order polarization (see eq. (1.2)) will thus be the most significant contributor to nonlinearity. Keeping in mind the definitions in eq. (1.2) and (1.15), we begin by expressing the time-domain integral as a convolution[8]:

$$\mathcal{P}_\mathrm{NL}(\mathbf{r},t) = 3\epsilon_0\chi^{(3)}_{xxxx}\mathcal{E}(\mathbf{r},t)\int \mathrm{d}\tau\, R(\tau)\left|\mathcal{E}(\mathbf{r},t-\tau)\right|^2 \quad (2.16a)$$
$$= 3\epsilon_0\chi^{(3)}_{xxxx}\mathcal{E}(\mathbf{r},t)\int \mathrm{d}\tau\, R(\tau)\mathcal{E}(\mathbf{r},t-\tau)\cdot\mathcal{E}^*(\mathbf{r},t-\tau) \quad (2.16b)$$
$$= 3\epsilon_0\chi^{(3)}_{xxxx}\mathcal{E}(\mathbf{r},t)\left[R(t)\otimes\left(\mathcal{E}(\mathbf{r},t)\cdot\mathcal{E}^*(\mathbf{r},t)\right)\right]. \quad (2.16c)$$

[7]It is implicit in this assumption that the spectral envelope does not change significantly when propagating over distance scales on the order of a wavelength.

[8]Since $\tilde{\mathcal{P}}_\mathrm{NL}(\mathbf{r},\omega)$ has been defined to have incorporated the shift of ω_0 in $\tilde{\mathcal{P}}^{(3)}$, we ignore the factor of $e^{-i\omega_0 t}$ here on purpose. The effect of this will be to simply give a Fourier transform that has no ω_0 term, and we take the shift to be implicit in the double integral specifying $\tilde{\mathcal{P}}_\mathrm{NL}(\mathbf{r},\omega)$ that is soon to follow.

Next, the Fourier transform of eq. (2.16c) is written as

$$\begin{aligned}\tilde{\mathcal{P}}_{\text{NL}}(\mathbf{r},\omega) &\equiv \mathscr{F}\left\{\mathcal{P}_{\text{NL}}(\mathbf{r},t)\right\}\\ &= \frac{3\epsilon_0\chi^{(3)}_{xxxx}}{2\pi}\tilde{\mathcal{E}}(\mathbf{r},\omega)\otimes\left[\mathscr{F}\left\{R(t)\otimes\left(\mathcal{E}(\mathbf{r},t)\cdot\mathcal{E}^*(\mathbf{r},t)\right)\right\}\right]. \quad (2.17)\end{aligned}$$

The transform in square brackets can be further manipulated to give

$$\tilde{\mathcal{P}}_{\text{NL}}(\mathbf{r},\omega) = \frac{3\epsilon_0\chi^{(3)}_{xxxx}}{2\pi}\tilde{\mathcal{E}}(\mathbf{r},\omega)\otimes\left[\tilde{R}(\omega)\mathscr{F}\left\{\left(\mathcal{E}(\mathbf{r},t)\mathcal{E}^*(\mathbf{r},t)\right)\right\}\right] \quad (2.18a)$$

$$= \frac{3\epsilon_0\chi^{(3)}_{xxxx}}{2\pi}\tilde{\mathcal{E}}(\mathbf{r},\omega)\otimes\left[\tilde{R}(\omega)\frac{\tilde{\mathcal{E}}(\mathbf{r},\omega)\otimes\tilde{\mathcal{E}}^*(\mathbf{r},\omega)}{2\pi}\right]. \quad (2.18b)$$

The next step is to expand the convolution term in square brackets, which takes the form

$$\tilde{R}(\omega)\frac{\tilde{\mathcal{E}}(\mathbf{r},\omega)\otimes\tilde{\mathcal{E}}^*(\mathbf{r},\omega)}{2\pi} = \tilde{R}(\omega)\int\frac{\mathrm{d}\omega''}{2\pi}\,\tilde{\mathcal{E}}(\mathbf{r},\omega'')\tilde{\mathcal{E}}^*(\mathbf{r},\omega-\omega''). \quad (2.19)$$

Now we can carry out the outer convolution, i.e. give the final form of the entire expression in eq. (2.18b):

$$\begin{aligned}\tilde{\mathcal{P}}_{\text{NL}}(\mathbf{r},\omega) = 3\epsilon_0\chi^{(3)}_{xxxx}\int\frac{\mathrm{d}\omega'}{2\pi}\,\tilde{\mathcal{E}}(\mathbf{r},\omega')\int\frac{\mathrm{d}\omega''}{2\pi}\,\big[\,&\tilde{\mathcal{E}}(\mathbf{r},\omega'')\tilde{\mathcal{E}}^*(\mathbf{r},\omega-\omega'-\omega'')\\ &\times\tilde{R}(\omega-\omega')\,\big]. \quad (2.20)\end{aligned}$$

Using the solution postulated in eq. (2.9), the term above becomes

$$\begin{aligned}\tilde{\mathcal{P}}_{\text{NL}}(\mathbf{r},\omega) = 3\epsilon_0\chi^{(3)}_{xxxx}\int\frac{\mathrm{d}\omega'}{2\pi}\int\frac{\mathrm{d}\omega''}{2\pi}\;\big[\,&\tilde{A}(z,\omega')F(\mathbf{r}_\perp,\omega')\tilde{A}(z,\omega'')F(\mathbf{r}_\perp,\omega'')\\ &\times\tilde{A}^*(z,\omega-\omega'-\omega'')F^*(\mathbf{r}_\perp,\omega-\omega'-\omega'')\tilde{R}(\omega-\omega')\\ &\times e^{i\beta(\omega')z}e^{i\beta(\omega'')z}e^{-i\beta(\omega-\omega'-\omega'')z}\,\big]. \quad (2.21)\end{aligned}$$

With the results above the complete form of the working equation becomes

$$
\begin{aligned}
F(\mathbf{r}_\perp,\omega)\frac{\partial \tilde{A}(z,\omega)}{\partial z} &= -\frac{\tilde{\alpha}(\omega)}{2}F(\mathbf{r}_\perp,\omega)\tilde{A}(z,\omega) + \frac{i\,3\epsilon_0\mu_0\chi^{(3)}_{xxxx}\omega^2 e^{-i\beta(\omega)z}}{2\beta(\omega)} \\
&\quad \times \int \frac{\mathrm{d}\omega'}{2\pi}\int \frac{\mathrm{d}\omega''}{2\pi}\ \Big[\ \tilde{A}(z,\omega')F(\mathbf{r}_\perp,\omega')\tilde{A}(z,\omega'')F(\mathbf{r}_\perp,\omega'') \\
&\quad \times \tilde{A}^*(z,\omega-\omega'-\omega'')F^*(\mathbf{r}_\perp,\omega-\omega'-\omega'')\tilde{R}(\omega-\omega') \\
&\quad \times e^{i\beta(\omega')z}e^{i\beta(\omega'')z}e^{-i\beta(\omega-\omega'-\omega'')z}\ \Big]\,.
\end{aligned}
\tag{2.22}
$$

To simplify the unwieldy expressions above, we make the following substitutions

$$
\tilde{A}'(z,\omega) \equiv \tilde{A}(z,\omega)e^{\frac{\tilde{\alpha}(\omega)z}{2}}, \tag{2.23a}
$$

$$
A(z,\omega) \equiv \tilde{A}(z,\omega)e^{i\beta(\omega)z}. \tag{2.23b}
$$

Recognizing that

$$
\begin{aligned}
F(\mathbf{r}_\perp,\omega)&\left[\frac{\partial \tilde{A}(z,\omega)}{\partial z}e^{\frac{\tilde{\alpha}(\omega)z}{2}} + \frac{\tilde{\alpha}(\omega)}{2}\tilde{A}(z,\omega)e^{\frac{\tilde{\alpha}(\omega)z}{2}}\right]e^{-\frac{\tilde{\alpha}(\omega)z}{2}} \\
&= e^{-\frac{\tilde{\alpha}(\omega)z}{2}}F(\mathbf{r}_\perp,\omega)\frac{\partial \tilde{A}'(z,\omega)}{\partial z},
\end{aligned}
\tag{2.24}
$$

eq. (2.22) becomes

$$
\begin{aligned}
F(\mathbf{r}_\perp,\omega)\frac{\partial \tilde{A}'(z,\omega)}{\partial z} &= \frac{i\,3\chi^{(3)}_{xxxx}\omega^2 e^{-i[\beta(\omega)+i\tilde{\alpha}(\omega)/2]z}}{2c^2\beta(\omega)}\int \frac{\mathrm{d}\omega'}{2\pi}\int \frac{\mathrm{d}\omega''}{2\pi}\ \Big[\ A(z,\omega')F(\mathbf{r}_\perp,\omega') \\
&\quad \times A(z,\omega'')F(\mathbf{r}_\perp,\omega'')A^*(z,\omega-\omega'-\omega'')F^*(\mathbf{r}_\perp,\omega-\omega'-\omega'') \\
&\quad \times \tilde{R}(\omega-\omega')\ \Big]\,.
\end{aligned}
\tag{2.25}
$$

It is useful to note that as a consequence of the substitutions in eqs. (2.23),

$$
A(z,\omega) = \tilde{A}'(z,\omega)e^{i\beta(\omega)z}e^{-\tilde{\alpha}(\omega)z/2} = \tilde{A}'(z,\omega)e^{i[\beta(\omega)+i\tilde{\alpha}(\omega)/2]z}, \tag{2.26}
$$

a relation that will be needed at a later stage.

The next step is to multiply both sides by $F^*(\mathbf{r}_\perp, \omega)$ and integrate over the transverse coordinates to get rid of the $\mathbf{r}_\perp$ dependence. The overlap integral involving F in the RHS of eq. (2.25) can be evaluated to first order as [17]

$$\int \mathrm{d}\mathbf{r}_\perp\, F^*(\mathbf{r}_\perp, \omega) F(\mathbf{r}_\perp, \omega') F(\mathbf{r}_\perp, \omega'') F^*(\mathbf{r}_\perp, \omega - \omega' - \omega'') \approx \left[V(\omega)V(\omega')V(\omega'')V(\omega - \omega' - \omega'')\right]^{1/4}, \quad (2.27)$$

where

$$V(\omega) \equiv \int \mathrm{d}\mathbf{r}_\perp \left|F(\mathbf{r}_\perp, \omega)\right|^4. \quad (2.28)$$

Moreover, the LHS will become

$$A_m(\omega) \frac{\partial \tilde{A}'(z, \omega)}{\partial z}, \quad (2.29)$$

with $A_m(\omega) \equiv \int \mathrm{d}\mathbf{r}_\perp \left|F(\mathbf{r}_\perp, \omega)\right|^2$.

Before presenting the result of the above operation on eq. (2.25), we first normalize the amplitude so that it's square represents the total power propagating across the fiber cross-section at a given longitudinal location z. For a given frequency ω, this power is (up to a factor of 2π)

$$\begin{aligned} \int \mathrm{d}\mathbf{r}_\perp \left|\tilde{\mathcal{E}}(\mathbf{r}, \omega)\right|^2 \times 2\epsilon_0 n_0 c &= \left|\tilde{A}'(z, \omega)\right|^2 \int \mathrm{d}\mathbf{r}_\perp \left|F(\mathbf{r}_\perp, \omega)\right|^2 \times 2\epsilon_0 n_0 c \\ &\equiv \left|\tilde{A}''(z, \omega)\right|^2, \end{aligned} \quad (2.30)$$

where the the normalized variable $\tilde{A}''(z, \omega)$ has been introduced. Enforcing this normalization means we must now use the expression

$$\tilde{A}'(z, \omega) = \frac{1}{\sqrt{2\epsilon_0 n_0 c}} \frac{\tilde{A}''(z, \omega)}{A_m^{1/2}(\omega)}, \quad (2.31a)$$

in eq. (2.25) for the amplitude. Likewise, we define

$$A(z, \omega) = \frac{1}{\sqrt{2\epsilon_0 n_0 c}} \frac{\underline{A}(z, \omega)}{A_m^{1/2}(\omega)}, \quad (2.31b)$$

keeping in mind that $\tilde{A}''(z,\omega) = \underline{A}(z,\omega)e^{-i[\beta(\omega)+i\tilde{\alpha}(\omega)/2]z}$, based on the relation in eq. (2.26).

We can now present the result of the above manipulations on eq. (2.25), which becomes

$$A_m(\omega)\frac{\partial}{\partial z}\frac{\tilde{A}''(z,\omega)}{\sqrt{A_m(\omega)}} = \frac{i3\chi^{(3)}_{xxxx}\omega^2 e^{-i\left[\beta(\omega)+i\tilde{\alpha}(\omega)/2\right]z}}{2c^2\beta(\omega)}\frac{1}{2\epsilon_0 n_0 c}\int\frac{\mathrm{d}\omega'}{2\pi}\int\frac{\mathrm{d}\omega''}{2\pi}\Bigg\{ \underline{A}(z,\omega')\underline{A}(z,\omega'')\underline{A}^*(z,\omega-\omega'-\omega'')\tilde{R}(\omega-\omega') \times\left[\frac{V(\omega)V(\omega')V(\omega'')V(\omega-\omega'-\omega'')}{A_m^2(\omega')A_m^2(\omega'')A_m^2(\omega-\omega'-\omega'')}\right]^{1/4}\Bigg\}. \tag{2.32}$$

The expression outside of the integral can be manipulated as follows

$$\frac{i3\chi^{(3)}_{xxxx}\omega^2}{2c^2\beta(\omega)}\frac{1}{2\epsilon_0 n_0 c} = i\frac{3\chi^{(3)}_{xxxx}\omega}{4\epsilon_0 n_0 c}\frac{\omega}{c\beta(\omega)}\frac{1}{c} = \frac{i n_0 n_2\omega}{c n_{\mathrm{eff}}(\omega)}, \tag{2.33}$$

where

$$n_{\mathrm{eff}}(\omega) \equiv \frac{c\beta(\omega)}{\omega} \quad \text{and} \quad n_2 \equiv \frac{3\chi^{(3)}_{xxxx}}{4\epsilon_0 n_0^2 c}. \tag{2.34}$$

Further, we make the definitions

$$A_{\mathrm{eff}}(\omega) \equiv \frac{\left[\int \mathrm{d}\mathbf{r}_\perp\left|F(\mathbf{r}_\perp,\omega)\right|^2\right]^2}{\int \mathrm{d}\mathbf{r}_\perp\left|F(\mathbf{r}_\perp,\omega)\right|^4} = \frac{A_m^2(\omega)}{V(\omega)}, \tag{2.35a}$$

$$\bar{A}(z,\omega) \equiv \frac{\underline{A}(z,\omega)}{A_{\mathrm{eff}}^{1/4}(\omega)}, \tag{2.35b}$$

$$\bar{\gamma}(\omega) \equiv \frac{n_0 n_2\omega}{c n_{\mathrm{eff}}(\omega)A_{\mathrm{eff}}^{1/4}(\omega)}, \tag{2.35c}$$

which simplifies eq. (2.32) to give

$$\frac{\partial\tilde{A}''(z,\omega)}{\partial z} = e^{-i\left[\beta(\omega)+i\tilde{\alpha}(\omega)/2\right]z}i\bar{\gamma}(\omega)\int\frac{\mathrm{d}\omega'}{2\pi}\int\frac{\mathrm{d}\omega''}{2\pi}\left[\bar{A}(z,\omega')\bar{A}(z,\omega'') \times \bar{A}^*(z,\omega-\omega'-\omega'')\tilde{R}(\omega-\omega')\right]. \tag{2.36}$$

It should be noted that the normalized amplitudes above represent the Fourier transforms of the longitudinal parts of the ansatz solution in eq. (2.9). We modify slightly our amplitudes here and include a factor of $e^{i\beta_0 z}$. This amounts to neglect of the absolute phase velocity since it ultimately leads to subtraction of β_0 from the dispersion operator. This can be used if the carrier-envelope phase is not a consideration. Further, to emphasize that we are looking at a right-sided spectrum, we make a small change in the amplitude frequency notation. This is accomplished as follows[9]:

$$\tilde{\omega} \equiv \omega - \omega_0, \tag{2.37a}$$

$$\underline{A}(z,\omega) \equiv \hat{A}(z,\tilde{\omega})e^{i\beta_0 z}. \tag{2.37b}$$

Keeping up with our previous definitions, we write

$$\tilde{A}''(z,\omega) = \underline{A}(z,\omega)e^{-i\left[\beta(\omega)+i\tilde{\alpha}(\omega)/2\right]z} = \hat{A}(z,\tilde{\omega})e^{-i\left[\beta(\omega)-\beta_0+i\tilde{\alpha}(\omega)/2\right]z}, \tag{2.38a}$$

$$\bar{A}(z,\omega) = \frac{\underline{A}(z,\omega)}{A_{\text{eff}}^{1/4}(\omega)} = \frac{\hat{A}(z,\tilde{\omega})e^{i\beta_0 z}}{A_{\text{eff}}^{1/4}(\omega)} \equiv \hat{\underline{A}}(z,\tilde{\omega})e^{i\beta_0 z}, \tag{2.38b}$$

$$\hat{\underline{A}}(z,t)e^{-i\omega_0 t} \equiv \mathscr{F}^{-1}\left\{\hat{\underline{A}}(z,\tilde{\omega})\right\}, \tag{2.38c}$$

$$\hat{A}(z,t) \equiv \mathscr{F}^{-1}\left\{\hat{A}(z,\omega)\right\}, \tag{2.38d}$$

where the last two equations specify the time-domain amplitudes that will be used in the steps to follow.

Now we make a transformation to a comoving time-frame by making the substitutions $T = t - \beta_1 z$ and $Z = z$ (see Appendix B), where

$$\beta_1 = \left.\frac{\mathrm{d}\beta(\omega)}{\mathrm{d}t}\right|_{\omega=\omega_0}, \tag{2.39}$$

and $1/\beta_1$ is the group velocity of the pulse. The effect of this will be to remove longitudinal motion from the pulse and to make it easier to focus on nonlinear effects.

[9]This step is unnecessary and can be left out. It is retained for completeness and so that the results can be easily matched with those in literature.

For this purpose we first expand the amplitude on the LHS of eq. (2.36) and express the integral on its RHS as a Fourier transform (just as the converse was done in the steps leading up to eq. (2.20)):

$$\frac{\partial \tilde{A}''(z,\omega)}{\partial z} = \left(\frac{\partial \hat{A}(z,\tilde{\omega})}{\partial z} - i\left[\beta(\omega)-\beta_0+i\tilde{\alpha}(\omega)/2\right]\hat{A}(z,\tilde{\omega})\right)e^{-i\left[\beta(\omega)-\beta_0+i\tilde{\alpha}(\omega)/2\right]z}$$
$$= e^{-i\left[\beta(\omega)-\beta_0+i\tilde{\alpha}(\omega)/2\right]z} i\bar{\gamma}(\omega)\mathscr{F}\left\{\underline{\hat{A}}(z,t)e^{-i\omega_0 t}\int \mathrm{d}\tau\, R(\tau)\left|\underline{\hat{A}}(z,t-\tau)\right|^2\right\}. \quad (2.40)$$

After canceling out the exponential term on each side, the important step here is to introduce a term involving β_1 so that the group velocity term cancels out in the time-domain when the transformation to a comoving frame is made. This is done by adding and subtracting a β_1 term in eq. (2.40) in the following manner:

$$\frac{\partial \hat{A}(z,\tilde{\omega})}{\partial z} - i\beta_1(\omega-\omega_0)\hat{A}(z,\tilde{\omega}) - i\left[\beta(\omega)-\beta_1(\omega-\omega_0)-\beta_0+i\tilde{\alpha}(\omega)/2\right]\hat{A}(z,\tilde{\omega})$$
$$= i\bar{\gamma}(\omega)\mathscr{F}\left\{\underline{\hat{A}}(z,t)e^{-i\omega_0 t}\int \mathrm{d}\tau\, R(\tau)\left|\underline{\hat{A}}(z,t-\tau)\right|^2\right\}. \quad (2.41)$$

We can now shift to time-domain by taking the inverse Fourier transform on both sides of the equation. The time-domain equation thus becomes

$$e^{-i\omega_0 t}\left(\frac{\partial \hat{A}(z,t)}{\partial z} + \beta_1\frac{\partial \hat{A}(z,t)}{\partial t}\right) - \mathscr{F}^{-1}\Big\{i\left[\beta(\omega)-\beta_1(\omega-\omega_0)-\beta_0+i\tilde{\alpha}(\omega)/2\right]$$
$$\times\ \hat{A}(z,\tilde{\omega})\Big\} = \mathscr{F}^{-1}\left\{i\bar{\gamma}(\omega)\mathscr{F}\left\{\underline{\hat{A}}(z,t)e^{-i\omega_0 t}\int \mathrm{d}\tau\, R(\tau)\left|\underline{\hat{A}}(z,t-\tau)\right|^2\right\}\right\}. \quad (2.42)$$

Going to comoving coordinates requires the transformations (see Appendix B)

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial T} \quad \text{and} \quad \frac{\partial}{\partial z} \to \frac{\partial}{\partial Z} - \beta_1\frac{\partial}{\partial T}, \quad (2.43)$$

which result in the following equation (note that the symbols for the amplitude variables have *not* been changed, though they are different now, because of the transformation to another time-frame):

$$e^{-i\omega_0 t}\left(\frac{\partial}{\partial Z}-\beta_1\frac{\partial}{\partial T}+\beta_1\frac{\partial}{\partial T}\right)\hat{A}(Z,T)$$
$$-\mathscr{F}^{-1}\left\{i\left[\beta(\omega)-\beta_1(\omega-\omega_0)-\beta_0+i\tilde{\alpha}(\omega)/2\right]\hat{A}(Z,\omega)\right\}$$
$$=\mathscr{F}^{-1}\left\{i\bar{\gamma}(\omega)\mathscr{F}\left\{\underline{\hat{A}}(Z,T)e^{-i\omega_0 T}\int \mathrm{d}\tau\, R(\tau)\left|\underline{\hat{A}}(Z,T-\tau)\right|^2\right\}\right\}. \quad (2.44)$$

Canceling out the $\beta_1\partial\hat{A}(Z,T)/\partial T$ terms at the beginning of the expression and moving to the frequency-domain again leaves us with[10]

$$\frac{\partial\hat{A}(Z,\tilde{\omega})}{\partial Z}-i\left[\beta(\omega)-\beta_1(\omega-\omega_0)-\beta_0+i\tilde{\alpha}(\omega)/2\right]\hat{A}(Z,\tilde{\omega})$$
$$=i\bar{\gamma}(\omega)\mathscr{F}_s\left\{\underline{\hat{A}}(Z,T)\int \mathrm{d}\tau\, R(\tau)\left|\underline{\hat{A}}(Z,T-\tau)\right|^2\right\}. \quad (2.45)$$

Cleaning things up a little bit, one version of the frequency-domain Generalized Nonlinear Schrödinger Equation (GNLSE) can be written as follows:

$$\frac{\partial\hat{A}'(Z,\tilde{\omega})}{\partial Z}=i\bar{\gamma}(\omega)\exp\left(-i\tilde{\beta}(\omega)Z\right)\mathscr{F}_s\left\{\underline{\hat{A}}(Z,T)\int \mathrm{d}\tau\, R(\tau)\left|\underline{\hat{A}}(Z,T-\tau)\right|^2\right\}, \quad (2.46a)$$

$$\hat{A}'(Z,\tilde{\omega})=\hat{A}(Z,\tilde{\omega})\exp\left(-i\tilde{\beta}(\omega)Z\right), \quad (2.46b)$$

$$\underline{\hat{A}}(Z,T)=\mathscr{F}^{-1}\left\{\frac{\hat{A}(Z,\tilde{\omega})}{A_{\mathrm{eff}}^{1/4}(\omega)}\right\}, \quad (2.46c)$$

$$\tilde{\beta}(\omega)\equiv\beta(\omega)-\beta_1(\omega-\omega_0)-\beta_0+i\tilde{\alpha}(\omega)/2, \quad (2.46d)$$

$$\bar{\gamma}(\omega)\equiv\frac{n_0 n_2\omega}{c n_{\mathrm{eff}}(\omega)A_{\mathrm{eff}}^{1/4}(\omega)}. \quad (2.46e)$$

2.1.1 Connection with time-domain GNLSE

The frequency-domain GNLSE presented in the last section transforms into the canonical time-domain GNLSE if the frequency dependence of the variables n_{eff}, A_{eff} and α

[10] $\mathscr{F}_s$ denotes the shifted Fourier transform defined by $\mathscr{F}_s\left\{A(z,t)\right\}=\int \mathrm{d}t\, A(z,t)e^{i(\omega-\omega_0)t}$.

is ignored (we take their values at the base frequency ω_0).

We simply write eq. (2.45) with $A_{\text{eff}}(\omega) \rightarrow A_{\text{eff}}(\omega_0)$, $n_{\text{eff}}(\omega) \rightarrow n_{\text{eff}}(\omega_0)$ and $\tilde{\alpha}(\omega) \rightarrow \alpha$ to get

$$\begin{aligned} \frac{\partial \hat{A}(Z,\tilde{\omega})}{\partial Z} &- i\left[\beta(\omega) - \beta_1(\omega-\omega_0) - \beta_0 + i\alpha/2\right]\hat{A}(Z,\tilde{\omega}) \\ &= i\,\frac{n_0 n_2 \omega}{c n_{\text{eff}}(\omega_0) A_{\text{eff}}^{1/4}(\omega_0)} \mathscr{F}_s\left\{\frac{\hat{A}(Z,T)}{A_{\text{eff}}^{1/4}(\omega_0)} \int \mathrm{d}\tau\, \frac{R(\tau)\left|\hat{A}(Z,T-\tau)\right|^2}{A_{\text{eff}}^{1/2}(\omega_0)}\right\}. \end{aligned} \tag{2.47}$$

This gives

$$\begin{aligned} \frac{\partial \hat{A}(Z,\tilde{\omega})}{\partial z} &- i\left[\beta(\omega) - \beta_1(\omega-\omega_0) - \beta_0\right]\hat{A}(Z,\tilde{\omega}) + i\frac{\alpha}{2}\hat{A}(Z,\tilde{\omega}) \\ &= i\tilde{\gamma}\left[1 + \frac{\tilde{\omega}}{\omega_0}\right] \mathscr{F}_s\left\{\hat{A}(Z,T)\int \mathrm{d}\tau\, R(\tau)\left|\hat{A}(Z,T-\tau)\right|^2\right\}, \end{aligned} \tag{2.48}$$

where

$$\tilde{\gamma} = \frac{n_0 n_2 \omega_0}{c n_{\text{eff}}(\omega_0) A_{\text{eff}}(\omega_0)}. \tag{2.49}$$

If we now take $n_{\text{eff}} \approx n_0$, write β as a Taylor expansion centered at ω_0 and operate on both sides by the shifted inverse Fourier transform the equation above can be seen to transform in time-domain to the oft-cited

$$\begin{aligned} \frac{\partial \hat{A}}{\partial T} + i\frac{\alpha}{2}\hat{A} - i\sum_{n=2}^{\infty}\frac{i^n\beta_n}{n!}\frac{\partial^n \hat{A}}{\partial T^n} &= i\gamma\left(1 + \frac{i}{\omega_0}\frac{\partial}{\partial T}\right) \\ &\times \left(\hat{A}(Z,T)\int \mathrm{d}\tau\, R(\tau)\left|\hat{A}(Z,T-\tau)\right|^2\right), \end{aligned} \tag{2.50}$$

where

$$\gamma \equiv \frac{n_2\omega_0}{cA_{\text{eff}}} \quad \text{and} \quad \beta_n \equiv \left.\frac{\mathrm{d}\beta(\omega)}{\mathrm{d}\omega}\right|_{\omega=\omega_0}. \tag{2.51}$$

CHAPTER 3

INTERMODAL FOUR-WAVE MIXING

3.1 Introduction

Four-wave (or four-photon) mixing is one of the nonlinear phenomena in optical fibers and results in coupling of energy between waves at different frequencies and modes [13]. Unsurprisingly it results from the third-order polarization, and a brief description can be given by considering the definition

$$\mathbf{P}^{(3)} = \epsilon_0 \chi^{(3)} \mathbf{EEE}. \tag{3.1}$$

It is clear from the three **E** terms that new frequencies being created will be from a sum of the frequencies of any three input waves. For example if we take

$$\mathbf{E}(\mathbf{r}, t) = \hat{\mathbf{p}} \sum_{n=1}^{3} E_n e^{i(\beta_n z - \omega_n)t} + \text{c.c.}, \tag{3.2}$$

then $\mathbf{P}^{(3)}$ will contain the term (assuming isotropy)

$$\hat{\mathbf{p}}\, 6\chi^{(3)}_{xxxx} E_1 E_2 E_3 \exp i \left[\left(\beta_1 + \beta_2 + \beta_3 \right) z - \left(\omega_1 + \omega_2 + \omega_3 \right) t \right].$$

It is this term we would focus on when considering third-harmonic generation, which involves the case of three photons transferring their energy to one photon with wavelength $\omega_4 = \omega_1 + \omega_2 + \omega_3$.

Another case would be $\omega_4 = 2\omega_1 - \omega_3$. This can be used to describe two photons at the pump wavelength ω_1 transferring their energy to one upshifted (anti-Stokes) photon at ω_4 and to one downshifted (Stokes) photon at ω_3, as shown in Figure 3.1 below. The relation above can be rearranged as $\omega_1 - \omega_3 = \omega_4 - \omega_1$ showing that the frequencies are detuned by equal amounts.

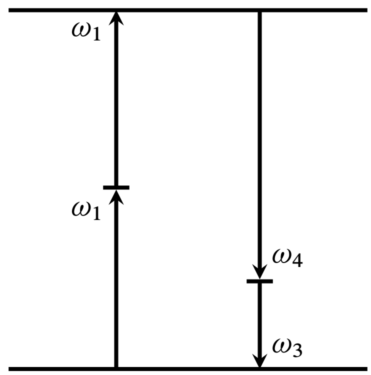

Figure 3.1. An illustration of Stokes and anti-Stokes generation that might result from four-wave mixing. The anti-Stokes photon has frequency ω_4 and the Stokes frequency is ω_3.

The associated phase-matching term is

$$\Delta\beta = 2\beta(\omega_1) - \beta(\omega_1 - \Delta\omega) - \beta(\omega_1 + \Delta\omega), \tag{3.3}$$

where $\Delta\omega$ is the absolute value of the frequency (up)downshift.

3.2 Extending the accuracy of the GNLSE

This section will be used to develop further the theory laid out in section 2.1. It will enable a more accurate description of how two (or possibly more) linearly polarized modes might interact in a higher-order-mode (HOM) fiber. Others have modeled multi-mode pulse propagation as well [20, 21], but with an approximate form of the effective area and using a time-domain approach.

Recall that after eq. (2.25) we made an approximation of the overlap integral using eq. (2.27). We now proceed from there in a more exact fashion. For convenience, eq. (2.25) is reproduced below.

$$F(\mathbf{r}_\perp,\omega)\frac{\partial \tilde{A}'(z,\omega)}{\partial z} = \frac{i3\chi^{(3)}_{xxxx}\omega^2 e^{-i[\beta(\omega)+i\tilde{\alpha}(\omega)/2]z}}{2c^2\beta(\omega)}\int\frac{d\omega'}{2\pi}\int\frac{d\omega''}{2\pi}\Big[\ A(z,\omega')F(\mathbf{r}_\perp,\omega')$$

$$\times A(z,\omega'')F(\mathbf{r}_\perp,\omega'')A^*(z,\omega-\omega'-\omega'')F^*(\mathbf{r}_\perp,\omega-\omega'-\omega'')$$

$$\times \tilde{R}(\omega-\omega')\ \Big].$$

Multiplying both sides by $F^*(\mathbf{r}_\perp,\omega)$, using the normalizations in eq. (2.31) and integrating over $\mathbf{r}_\perp$ now gives us

$$A_m(\omega)\frac{\partial}{\partial z}\frac{\tilde{A}''(z,\omega)}{\sqrt{A_m(\omega)}} = \frac{i3\chi^{(3)}_{xxxx}\omega^2 e^{-i[\beta(\omega)+i\tilde{\alpha}(\omega)/2]z}}{2c^2\beta(\omega)}\frac{1}{2\epsilon_0 n_0 c}\int\frac{d\omega'}{2\pi}\int\frac{d\omega''}{2\pi}\Bigg\{$$

$$\underline{A}(z,\omega')\underline{A}(z,\omega'')\underline{A}^*(z,\omega-\omega'-\omega'')\tilde{R}(\omega-\omega')$$

$$\times\left[\int d\mathbf{r}_\perp \frac{F^*(\mathbf{r}_\perp,\omega)F(\mathbf{r}_\perp,\omega')F(\mathbf{r}_\perp,\omega'')F^*(\mathbf{r}_\perp,\omega-\omega'-\omega'')}{A_m^{1/2}(\omega')A_m^{1/2}(\omega'')A_m^{1/2}(\omega-\omega'-\omega'')}\right]\Bigg\}, \quad (3.5)$$

where, as before

$$A_m(\omega) \equiv \int d\mathbf{r}_\perp \left|F(\mathbf{r}_\perp,\omega)\right|^2.$$

We now normalize the transverse mode profile to $\underline{F}$ so that the integral over $\mathbf{r}_\perp$ above is simplified. This is accomplished by the definition

$$\underline{F}(\mathbf{r}_\perp,\omega) \equiv \frac{F(\mathbf{r}_\perp,\omega)}{A_m^{1/2}(\omega)}, \quad (3.6)$$

from which it is clear that

$$\int d\mathbf{r}_\perp \left|\underline{F}(\mathbf{r}_\perp,\omega)\right|^2 = 1.$$

Taking $\underline{A}(z,\omega) \equiv \hat{\underline{A}}(z,\omega)e^{i\beta_0 z}$ (for simplicity we are ignoring notation that uses the shift by ω_0 here) the equation becomes

$$\frac{\partial \hat{A}''(z,\omega)}{\partial z} = i\tilde{\gamma}(\omega)e^{-i[\beta(\omega)-\beta_0+i\tilde{\alpha}(\omega)/2]z}\int d\mathbf{r}_\perp \underline{F}^*(\mathbf{r}_\perp,\omega)\int\frac{d\omega'}{2\pi}\int\frac{d\omega''}{2\pi}\Big[$$

$$\times\hat{\underline{A}}(z,\omega')\underline{F}(\mathbf{r}_\perp,\omega')\hat{\underline{A}}(z,\omega'')\underline{F}(\mathbf{r}_\perp,\omega'')\tilde{R}(\omega-\omega')$$

$$\times\hat{\underline{A}}^*(z,\omega-\omega'-\omega'')\underline{F}^*(\mathbf{r}_\perp,\omega-\omega'-\omega'')\ \Big]. \quad (3.7)$$

The transformation to a comoving time-frame can be done just as before. In short we have $\beta(\omega) \to \beta(\omega) - \beta_1\omega$ and the working equation, after carrying out the same steps now appears as

$$\frac{\partial \hat{A}(Z,\omega)}{\partial Z} - i\left[\beta(\omega) - \beta_1\omega - \beta_0 + i\tilde{\alpha}(\omega)/2\right]\hat{A}(Z,\omega) = \\ i\tilde{\gamma}(\omega)\int \mathrm{d}\mathbf{r}_\perp \underline{F}^*(\mathbf{r}_\perp,\omega)\int\frac{\mathrm{d}\omega'}{2\pi}\int\frac{\mathrm{d}\omega''}{2\pi}\Big[\tilde{R}(\omega-\omega')\tilde{H}(z,\omega')\tilde{H}(z,\omega'') \\ \times \tilde{H}^*(z,\omega-\omega'-\omega'')\Big]. \quad (3.8)$$

In the relation above, $\tilde{H}(\mathbf{r},\omega) \equiv \underline{\hat{A}}(z,\omega)\underline{F}(\mathbf{r}_\perp,\omega)$, and the definitions in eqs. (2.38) still hold. A comparison with eq. (2.40) is instructive, and we notice that the main difference between the two lies in the integrals.

3.3 Verification of the extended model

The ideas introduced in section 3.2 can be used to describe the results of an experiment [22] where a degenerated pump pulse at wavelength 1064 nm in the LP_{01} mode generated highly efficient four-wave mixing with a frequency detuning of 35.2 THz. The measured output spectrum is shown in Figure 3.2. The pulsed output was achieved using a mode-locked fiber laser with a repetition rate of 18.33 MHz, approximate pulse width of 6 ps and average peak power of 470 mW. A higher-order-mode fiber of length 1.75 m was spliced directly to the laser and the output spectrum for a pulse of energy 21 nJ was recorded.

In this case we have a signal (the lower energy Stokes band) in the LP_{02} mode around 1200 nm and an idler (anti-Stokes wave) centered at 940 nm in the LP_{01} mode. The phase-matching term becomes

$$\Delta\beta = 2\beta_{01}(\omega_\mathrm{p}) - \beta_{01}(\omega_\mathrm{p} - \Delta\omega) - \beta_{02}(\omega_\mathrm{p} + \Delta\omega), \quad (3.9)$$

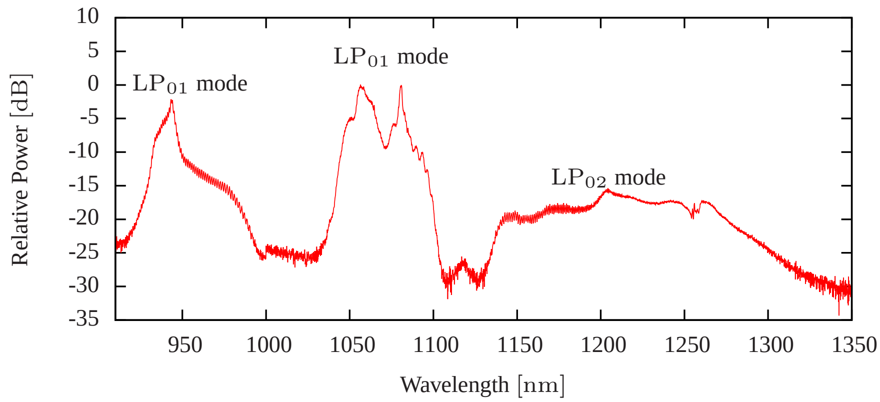

Figure 3.2. The output spectrum measured in the experiment [1].

and it can be numerically calculated to confirm the detuning observed ($\Delta\beta$ goes to zero at the experimentally observed detuning in [22]).

The four-wave mixing transverse field overlap integral $A_{\text{eff, FWM}}$ defined as

$$A^{-1}_{\text{eff, FWM}}(\Delta\omega) \equiv \int \mathrm{d}\mathbf{r}_\perp \Big[\underline{F}^*_{01}(\mathbf{r}_\perp, \omega_\text{p}) \underline{F}_{01}(\mathbf{r}_\perp, \omega_\text{p}) \underline{F}_{01}(\mathbf{r}_\perp, \omega_\text{p} + \Delta\omega) \times \underline{F}^*_{02}(\mathbf{r}_\perp, \omega_\text{p} - \Delta\omega) \Big], \tag{3.10}$$

can be used to calculate the correct effective area under phase-matching. This has been done in [1] to show that the use of the cross-mode effective area A_{eff} which is defined as

$$\bar{A}^{-1}_{\text{eff}}(\omega) \equiv \int \mathrm{d}\mathbf{r}_\perp \underline{F}^*_A(\mathbf{r}_\perp, \omega) \underline{F}_B(\mathbf{r}_\perp, \omega) \underline{F}_C(\mathbf{r}_\perp, \omega) \underline{F}^*_D(\mathbf{r}_\perp, \omega). \tag{3.11}$$

is invalid.

A_{eff} ignores the convolution of the transverse modes inside the integral involving the nonlinear polarization, which is inaccurate. The value of $A_{\text{eff, FWM}}$ turns out to be relatively small ($7\,\mu\text{m}^2$), which indicates that the transverse overlap integral is significant, and that the four-wave mixing process is efficient. The quarter-root implementation also turns out to be inadequate [1].

Due to the issues mentioned above, the general model in section 3.2 needs to be employed. This calls for the use of eq. (3.8). For a given mixing combination, we write the integral on its RHS as

$$i\tilde{\gamma}(\omega)\int \mathrm{d}\mathbf{r}_\perp \underline{F}_A^*(\mathbf{r}_\perp,\omega)\int \frac{\mathrm{d}\omega'}{2\pi}\int \frac{\mathrm{d}\omega''}{2\pi}\left[\tilde{R}(\omega-\omega')\tilde{H}_B(\mathbf{r},\omega')\tilde{H}_C^*(\mathbf{r},\omega-\omega'-\omega'') \times \tilde{H}_D(\mathbf{r},\omega'')\right]. \tag{3.12}$$

The frequency integral above can be written in the form of nested Fourier transforms, which could be useful in computation. To see this, note that it can be expressed by the convolution

$$\mathscr{F}\left\{H_B\left[R\otimes\left(H_D H_C^*\right)\right]\right\} = \mathscr{F}\left\{\mathscr{F}^{-1}\left\{\tilde{H}_B\right\}\mathscr{F}^{-1}\left\{\mathscr{F}\left\{\left[R\otimes\left(H_D H_C^*\right)\right]\right\}\right\}\right\}, \tag{3.13}$$

just as eq. (2.16c) was used to obtain eq. (2.20). Noting that the time-domain expressions have no tilde above them, the inner Fourier transform can be manipulated further to give

$$\begin{aligned}\mathscr{F}\left\{\left[R\otimes\left(H_D H_C^*\right)\right]\right\} &= \mathscr{F}\{R\}\,\mathscr{F}\left\{H_D H_C^*\right\}\\ &= \tilde{R}\,\mathscr{F}\left\{\mathscr{F}^{-1}\left\{\tilde{H}_D\right\}\mathscr{F}^{-1}\left\{\tilde{H}_C^*\right\}\right\},\end{aligned} \tag{3.14}$$

so that the total expression in eq (3.13) reads

$$\mathscr{F}\left\{\mathscr{F}^{-1}\left\{\tilde{H}_B\right\}\mathscr{F}^{-1}\left\{\tilde{R}\,\mathscr{F}\left\{\mathscr{F}^{-1}\left\{\tilde{H}_D\right\}\mathscr{F}^{-1}\left\{\tilde{H}_C^*\right\}\right\}\right\}\right\}. \tag{3.15}$$

This means that the integral we started with in eq. (3.12) is now

$$\tilde{Q}_{A,B,C,D} \equiv i\tilde{\gamma}(\omega)\int \mathrm{d}\mathbf{r}_\perp \underline{F}_A^*(\mathbf{r}_\perp,\omega)\mathscr{F}\left\{\mathscr{F}^{-1}\left\{\tilde{H}_B\right\}\mathscr{F}^{-1}\left\{\tilde{R}\,\mathscr{F}\left\{\mathscr{F}^{-1}\left\{\tilde{H}_D\right\}\mathscr{F}^{-1}\left\{\tilde{H}_C^*\right\}\right\}\right\}\right\}. \tag{3.16}$$

By including the different mixing combinations (described in section 3.1) between the two modes in the experiment, the extended GNLSEs for both of them can now be written as

$$\frac{\partial \hat{A}_{01}(Z,\omega)}{\partial Z} - i\left[\beta_{01}(\omega) - \beta_1^{\text{ref}}\omega - \beta_0^{\text{ref}} + i\tilde{\alpha}_{01}(\omega)/2\right]\hat{A}_{01}(Z,\omega) = \sum_{B,C,D\in\{01,02\}} \tilde{Q}_{01,B,C,D}\,, \tag{3.17a}$$

$$\frac{\partial \hat{A}_{02}(Z,\omega)}{\partial Z} - i\left[\beta_{02}(\omega) - \beta_1^{\text{ref}}\omega - \beta_0^{\text{ref}} + i\tilde{\alpha}_{02}(\omega)/2\right]\hat{A}_{02}(Z,\omega) = \sum_{B,C,D\in\{01,02\}} \tilde{Q}_{02,B,C,D}. \tag{3.17b}$$

β_1^{ref} is the inverse group velocity for the reference mode and β_0^{ref} is its propagation constant. Each summation when expanded will consist of eight terms, each of which corresponds to a specific four-wave mixing process between the interacting modes. The details of this are covered more thoroughly in the next section.

Use of the coupled-mode equations above has yielded results that match experimental data well [1]. It turns out that there do exist some other phase-matched four-wave mixing processes as well, but the dominant one is the $\{3 \times \mathrm{LP}_{01},\ 1 \times \mathrm{LP}_{02}\}$ process.

3.4 Extension to unidirectional field-vector equations

While the work presented hitherto has treated linearly polarized modes, Maxwell's equations can be used to arrive at a more general propagation equation. A detailed derivation in [23] that assumes fields of the form (compare with eq. (2.9))

$$\mathbf{E}(\mathbf{r},t) = \sum_m \int \frac{\mathrm{d}\omega}{2\pi} \hat{A}_m(z,\omega)\tilde{\mathbf{E}}_m(\mathbf{r}_\perp,\omega)e^{i(\hat{\beta}_m(\omega)z-\omega t)}, \tag{3.18a}$$

$$\mathbf{H}(\mathbf{r},t) = \sum_m \int \frac{\mathrm{d}\omega}{2\pi} \hat{A}_m(z,\omega)\tilde{\mathbf{H}}_m(\mathbf{r}_\perp,\omega)e^{i(\hat{\beta}_m(\omega)z-\omega t)}, \tag{3.18b}$$

where the sum is over the fiber modes m, leads to

$$\frac{\partial \hat{A}_m(z,\omega)}{\partial z} = \frac{i\omega e^{-i\hat{\beta}_m(\omega)z}}{N_m(\omega)} \int \mathrm{d}\mathbf{r}_\perp \tilde{\mathbf{E}}_m^*(\mathbf{r}_\perp,\omega)\cdot\tilde{\mathbf{P}}_{\mathrm{NL}}(\mathbf{r},\omega), \tag{3.19a}$$

$$\begin{aligned}\delta_{m,n}N_m(\omega) \equiv \int \mathrm{d}\mathbf{r}_\perp \hat{\mathbf{z}}\cdot\Big(&\tilde{\mathbf{E}}_m(\mathbf{r}_\perp,\omega)\times\tilde{\mathbf{H}}_n^*(\mathbf{r}_\perp,\omega) \\ &-\tilde{\mathbf{H}}_m(\mathbf{r}_\perp,\omega)\times\tilde{\mathbf{E}}_n^*(\mathbf{r}_\perp,\omega)\Big).\end{aligned} \tag{3.19b}$$

Assuming isotropy, like in eq. (2.16c) the nonlinear polarization can be specified as[1]

$$\mathbf{P}_{\mathrm{NL}}(\mathbf{r},t) = \epsilon_0\chi^{(3)}_{xxxx}\mathbf{E}(\mathbf{r},t)\Big[R(t)\otimes\big(\mathbf{E}(\mathbf{r},t)\cdot\mathbf{E}^*(\mathbf{r},t)\big)\Big].$$

The solutions above can be further expanded by following steps that are nearly identical to the single-mode treatment, with $\tilde{\mathbf{P}}_{\mathrm{NL}}(\mathbf{r},\omega)$ (the Fourier transform of $\mathbf{P}_{\mathrm{NL}}(\mathbf{r},t)$) being written as

$$\frac{\epsilon_0\chi^{(3)}_{xxxx}}{2\pi}\tilde{\mathbf{E}}(\mathbf{r},\omega)\otimes\Big[\tilde{R}(\omega)\mathscr{F}\Big\{\big(\mathbf{E}(\mathbf{r},t)\cdot\mathbf{E}^*(\mathbf{r},t)\big)\Big\}\Big] \tag{3.20a}$$

$$\begin{aligned}= \frac{\epsilon_0\chi^{(3)}_{xxxx}}{2\pi}\begin{bmatrix}\tilde{E}_x(\mathbf{r},\omega) & \tilde{E}_y(\mathbf{r},\omega) & \tilde{E}_z(\mathbf{r},\omega)\end{bmatrix}^T \otimes \Big(\tilde{R}(\omega)\mathscr{F}\big\{&E_x(\mathbf{r},t)E_x^*(\mathbf{r},t) \\ &+E_y(\mathbf{r},t)E_y^*(\mathbf{r},t)+E_z(\mathbf{r},t)E_z^*(\mathbf{r},t)\big\}\Big).\end{aligned} \tag{3.20b}$$

The expression in round brackets above now becomes

$$\tilde{R}(\omega)\int\frac{\mathrm{d}\omega''}{2\pi}\tilde{\mathbf{E}}(\mathbf{r},\omega'')\cdot\tilde{\mathbf{E}}^*(\mathbf{r},\omega-\omega''), \tag{3.21}$$

[1]Note that the field and polarization definitions here are different from those in eq. (1.14).

and the final convolution (similar to the one used to obtain eq. (2.20)) gives

$$\tilde{\mathbf{P}}_{\rm NL}(\mathbf{r},\omega) = \epsilon_0 \chi^{(3)}_{xxxx} \int \frac{\mathrm{d}\omega'}{2\pi} \tilde{\mathbf{E}}(\mathbf{r},\omega') \int \frac{\mathrm{d}\omega''}{2\pi} \Big[\tilde{\mathbf{E}}(\mathbf{r},\omega'') \cdot \tilde{\mathbf{E}}^*(\mathbf{r},\omega-\omega'-\omega'') \\ \times \tilde{R}(\omega-\omega') \Big]. \tag{3.22}$$

Noting that the full electric field in eq. (3.18) is a sum over the modal fields and using the nonlinear polarization in eq. (3.19a) results in the following propagation equation:

$$\frac{\partial \hat{A}_m(z,\omega)}{\partial z} = \frac{i\epsilon_0 \chi^{(3)}_{xxxx} \omega e^{-i\hat{\beta}_m(\omega)z}}{N_m(\omega)} \sum_{npq} \int \mathrm{d}\mathbf{r}_\perp \int \frac{\mathrm{d}\omega'}{2\pi} \int \frac{\mathrm{d}\omega''}{2\pi} \Big[\\ \hat{A}_n(z,\omega')\hat{A}_p(z,\omega'')\hat{A}^*_q(z,\omega-\omega'-\omega'') \\ \times \tilde{\mathbf{E}}^*_m(\mathbf{r}_\perp,\omega)\cdot\tilde{\mathbf{E}}_n(\mathbf{r}_\perp,\omega')\tilde{\mathbf{E}}_p(\mathbf{r}_\perp,\omega'')\cdot\tilde{\mathbf{E}}^*_q(\mathbf{r}_\perp,\omega-\omega'-\omega'') \\ \times e^{i\hat{\beta}_n(\omega')z} e^{i\hat{\beta}_p(\omega'')z} e^{-i\hat{\beta}_q(\omega-\omega'-\omega'')z} \tilde{R}(\omega-\omega') \Big], \tag{3.23}$$

where the summation involving n, p and q is over all possible modes. Eq. (3.23) is the general frequency-domain equation that governs mode propagation in multimode fibers. Simplifications to it can be made by finding approximations for the overlap integrals.

We can also show that for a single mode, eq. (3.23) reduces to the formulation developed in earlier chapters. Before that, a slight modification in notation will be made to reconcile the difference in field definitions in earlier sections and this one. Note that since the fields in eq. (3.19a) are real, $\hat{\beta}(-\omega) = -\hat{\beta}(\omega)$, $\hat{A}_m(z,-\omega) = \hat{A}^*_m(z,\omega)$ and $\tilde{\mathbf{E}}_m(\mathbf{r}_\perp,-\omega) = \tilde{\mathbf{E}}^*_m(\mathbf{r}_\perp,\omega)$ (ditto for $\tilde{\mathbf{H}}(\mathbf{r}_\perp,\omega)$) . Assuming a scalar single-mode field we may write[2]

$$\hat{A}(z,\omega)\tilde{E}(\mathbf{r}_\perp,\omega)e^{i\hat{\beta}(\omega)z} = \tilde{A}(z,\omega)F(\mathbf{r}_\perp,\omega)e^{i\beta(\omega)z} + \text{c.c.}, \tag{3.24a}$$

$$\hat{A}(z,\omega)\tilde{H}(\mathbf{r}_\perp,\omega)e^{i\hat{\beta}(\omega)z} = \tilde{A}(z,\omega)H(\mathbf{r}_\perp,\omega)e^{i\beta(\omega)z} + \text{c.c.} \tag{3.24b}$$

[2]Assuming the envelopes are separated enough.

The electromagnetic field amplitudes are related as follows:

$$H(\mathbf{r}_\perp, \omega) = \frac{1}{\mu_0} \frac{\beta(\omega)}{\omega} F(\mathbf{r}_\perp, \omega). \tag{3.25}$$

Taking **H** and **E** to be orthogonally polarized with respect to each other we calculate N as[3]

$$N(\omega) = \int \mathrm{d}\mathbf{r}_\perp \left(\left| F(\mathbf{r}_\perp, \omega) \right|^2 + \left| F(\mathbf{r}_\perp, \omega) \right|^2 \right) \frac{1}{\mu_0} \frac{\beta(\omega)}{\omega} \tag{3.26a}$$

$$= \frac{2}{\mu_0} \frac{\beta(\omega)}{\omega} A_m(\omega). \tag{3.26b}$$

With the above changes, eq. (3.23) becomes

$$\begin{aligned} \frac{\partial \tilde{A}(z, \omega)}{\partial z} = \frac{i 3 \epsilon_0 \mu_0 \chi^{(3)}_{xxxx} \omega^2 e^{-i\beta(\omega) z}}{2\beta(\omega) A_m(\omega)} \int \mathrm{d}\mathbf{r}_\perp \int \frac{\mathrm{d}\omega'}{2\pi} \int \frac{\mathrm{d}\omega''}{2\pi} \Big[& \\ \tilde{A}(z, \omega') \tilde{A}(z, \omega'') \tilde{A}^*(z, \omega - \omega' - \omega'') & \\ \times F^*(\mathbf{r}_\perp, \omega) F(\mathbf{r}, \omega') F(\mathbf{r}_\perp, \omega'') F^*(\mathbf{r}_\perp, \omega - \omega' - \omega'') & \\ \times e^{i\beta(\omega') z} e^{i\beta(\omega'') z} e^{-i\beta(\omega - \omega' - \omega'') z} \tilde{R}(\omega - \omega') \Big]. & \end{aligned} \tag{3.27}$$

If we recall that $A_m(\omega) \equiv \int \mathrm{d}\mathbf{r}_\perp \left| F(\mathbf{r}_\perp, \omega) \right|^2$ – then, barring the absorption term – the equation above is precisely equal to eq. (2.22) after the latter has been multiplied by $F^*(\mathbf{r}_\perp, \omega)$ and integrated over the transverse coordinates.

[3]Note that the subscript in A_m has nothing to do with the mode m since this is the single-mode case.

APPENDIX A

FOURIER TRANSFORM DEFINITIONS

The Fourier transform pair is defined in most mathematical texts as

$$G(s) = \int \mathrm{d}t\, g(t) e^{-2\pi i s t} \qquad\qquad g(t) = \int \mathrm{d}s\, G(s) e^{2\pi i s t}. \tag{A.1}$$

With $\omega = 2\pi s$ and a bit of manipulation, it is easy to show that the definition

$$F(\omega) = \int \mathrm{d}t\, f(t) e^{i\omega t} \qquad\qquad f(t) = \int \frac{\mathrm{d}\omega}{2\pi} F(\omega) e^{-i\omega t}. \tag{A.2}$$

is equivalent. This is the form used in this document, with the operator definition $\mathscr{F}\left\{f(t)\right\} = F(\omega)$ and $\mathscr{F}^{-1}\left\{F(\omega)\right\} = f(t)$.

The transform of the product is the convolution of the individual transforms. To see this write

$$\begin{aligned}\mathscr{F}\left\{f(t)h(t)\right\} &= \int \mathrm{d}t\, e^{i\omega t} \int \frac{\mathrm{d}\omega''}{2\pi} F(\omega'') e^{-i\omega'' t} \int \frac{\mathrm{d}\omega'}{2\pi} H(\omega') e^{-i\omega' t} \\ &= \int \mathrm{d}t \int \frac{\mathrm{d}\omega'}{2\pi} \int \frac{\mathrm{d}\omega''}{2\pi} F(\omega'') H(\omega') e^{i(\omega-\omega'-\omega'')t}. \end{aligned} \tag{A.3a}$$

Noting that

$$\int \mathrm{d}t\, e^{i(\omega-\omega'-\omega'')t} = 2\pi\delta(\omega-\omega'-\omega''), \tag{A.4}$$

and integrating over ω'' gives

$$\int \frac{\mathrm{d}\omega'}{2\pi} H(\omega') F(\omega-\omega') = \frac{1}{2\pi} H(\omega) \otimes F(\omega) = \mathscr{F}\left\{f(t)h(t)\right\}. \tag{A.5}$$

A similar procedure can be used to show that

$$\mathscr{F}^{-1}\left\{F(\omega)G(\omega)\right\} = f(t) \otimes h(t). \tag{A.6}$$

These results have been used extensively throughout this document. In Section 1.4 for example, we use

$$\begin{aligned}
\mathscr{F}\left\{A(0,t)e^{-i\omega_0 t}\right\} &= \frac{1}{2\pi}\tilde{A}(0,\omega)\otimes\mathscr{F}\left\{e^{-i\omega_0 t}\right\} \\
&= \frac{1}{2\pi}\int \mathrm{d}\omega'\,\tilde{A}(0,\omega)2\pi\delta(\omega-\omega_0-\omega') \\
&= \tilde{A}(0,\omega-\omega_0). \qquad \text{(A.7)}
\end{aligned}$$

APPENDIX B

TRANSFORMATION TO LOCAL TIME COORDINATES

Let the time-domain envelope be $A(z,t) = u(Z,T)$ with $Z = z$ and $T = t - \beta_1 z$. Then

$$\frac{\partial A}{\partial t} = \frac{\partial}{\partial z} u(Z, t - \beta_1 z) = \frac{\partial}{\partial t} u\left(Z, T(z)\right) = \frac{\partial u}{\partial Z} \overbrace{\frac{\partial Z}{\partial t}}^{=0} + \frac{\partial u}{\partial T} \overbrace{\frac{\partial T}{\partial t}}^{=1} = \frac{\partial u}{\partial T}. \quad \text{(B.1a)}$$

$$\frac{\partial A}{\partial z} = \frac{\partial u}{\partial Z} \overbrace{\frac{\partial Z}{\partial z}}^{=1} + \frac{\partial u}{\partial T} \overbrace{\frac{\partial T}{\partial z}}^{-\beta_1} = \frac{\partial u}{\partial Z} - \beta_1 \frac{\partial u}{\partial T}. \quad \text{(B.1b)}$$

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